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Mathematical tools for the Physical Sciences

- Math 191 A.

Lecture 1.

24 Aug 2016.

Series: The problem we are interested in is that of ~~ever~~ adding infinitely many real numbers. This is a problem naturally arising in a range of situations.

For instance:

Example:

With my 1st step I cover distance $\frac{1}{2}$.

" " 2nd step " " $\frac{1}{2}$ of the distance I covered in Step 1.

" " 3rd step " " $\frac{1}{2}$ of the distance I covered in Step 2,

:

How much distance will I cover in total?

Well, at the n -th step I cover distance $\frac{1}{2^n}$,

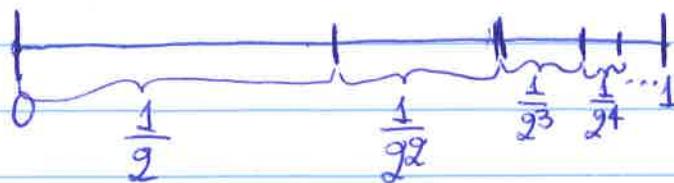
so in total I cover distance

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

When I start putting these lengths consecutively

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on a line segment of length 1,



it is easy to see that, every time, I add actually half the length of what remains of the interval $[0, 1]$

(because what remains is exactly equal to the distance I covered in the last step).

Thanks to this fact (special for this case), we can guess that the distance I will cover eventually is 1; the reason is that $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}$ gets closer and closer to 1 as n gets larger and larger.

This guess is indeed correct, i.e.

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = 1. \text{ This is not a proper proof though...}$$

Let's change things a little in this example: suppose that I cover distance $\frac{1}{3}$ in the 1st step,

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and, at each step after that, I cover $\frac{1}{3}$ of

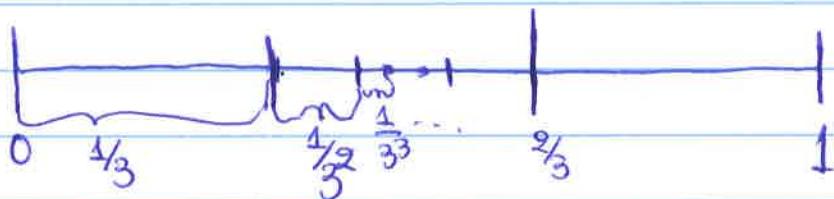
the distance I covered in the last step. Then,

at the n -th step I will cover distance $\frac{1}{3^n}$,

so in total I will cover distance

$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$. Let's cover the interval

$[0,1]$ consecutively with these lengths:



One can see that $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$ will be

a point on the left of $\frac{2}{3}$ in $[0,1]$;

so, we can guess that $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$

will be some specific number, but it is harder to guess what that number will be.

We therefore need to find an efficient way

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to calculate $\sum_{k=1}^{+\infty} a_k$, for $a_k \in \mathbb{R}$.

$$\underbrace{\dots}_{\parallel} \quad \sum_{k=1}^{+\infty} a_k$$

$a_1 + a_2 + \dots + a_n + \dots$, a series.

By the above example, it feels that the infinite sum $\sum_{k=1}^{+\infty} a_k$ should behave like

$a_1 + a_2 + \dots + a_n$, for very large n . This is indeed

true; it is actually ~~is~~ this way that we

define $\sum_{k=1}^{+\infty} a_k$: $a_1 + a_2 + \dots$ is the $\lim_{n \rightarrow +\infty} (a_1 + a_2 + \dots + a_n)$
(which may not exist):

Definition: Let $(a_k)_{k \in \mathbb{N}}$ be a sequence.

Let $s_n := a_1 + \dots + a_n$, $\forall n \in \mathbb{N}$.

We call s_n the n -th partial sum of the series $\sum_{k=1}^{+\infty} a_k$.

- If $(s_n)_{n \in \mathbb{N}}$ converges to some $s \in \mathbb{R}$,

we say that $\sum_{k=1}^{+\infty} a_k$ converges to s ,

and we write $\sum_{k=1}^{+\infty} a_k = s$. (this means that $a_1 + a_2 + \dots = s$)

- If $(s_n)_{n \in \mathbb{N}}$ does not converge, then

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we say that

$$\sum_{k=1}^{+\infty} a_k$$

diverges.

- In particular: If $s_n \xrightarrow{n \rightarrow +\infty} +\infty$ (or $-\infty$),

~~this means
that
 $a_1 + a_2 + \dots$
 $= +\infty$ (or $-\infty$)~~

then we say that $\sum_{k=1}^{+\infty} a_k$ diverges to $+\infty$
(or $-\infty$) and we write $\sum_{k=1}^{+\infty} a_k = +\infty$
(or $-\infty$).

→ In the case where $\lim_{n \rightarrow +\infty} s_n$

doesn't exist (in \mathbb{R} or $\{-\infty, +\infty\}$),
then we cannot calculate
 $\sum_{k=1}^{+\infty} a_k$. I.e., $a_1 + a_2 + \dots$ doesn't
make sense.

ex: • $a_k = 1$, $\forall k \in \mathbb{N}$. What is $\sum_{k=1}^{+\infty} a_k$?

Answer: $s_n = a_1 + a_2 + \dots + a_n = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n$,

so $s_n \xrightarrow{n \rightarrow +\infty} +\infty$, so $\sum_{k=1}^{+\infty} a_k$

is a divergent series, with $\sum_{k=1}^{+\infty} a_k = +\infty$.

• $a_k = (-1)^k$, $\forall k \in \mathbb{N}$. What is $\sum_{k=1}^{+\infty} a_k$?

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Answer: $\alpha_1 = -1, \alpha_2 = 1, \alpha_3 = -1, \alpha_4 = 1, \dots$

So, $s_1 = -1, s_2 = 0, s_3 = -1, s_4 = 0, \dots$

$$s_n = \begin{cases} -1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \quad \text{So, } \lim_{n \rightarrow \infty} s_n$$

doesn't exist (in $\mathbb{R} \cup \{\pm\infty\}$), so
 $\sum_{k=1}^{+\infty} \alpha_k$ is a divergent series, and

$\alpha_1 + \alpha_2 + \dots$ doesn't make sense.

- Geometric series: $\sum_{k=1}^{+\infty} x^k$, for some fixed $x \in \mathbb{R}$.

We just considered the cases for $x=1$ and $x=-1$, and showed that $\sum_{k=1}^{+\infty} x^k$ diverges

in these situations. Earlier, we considered $x = \frac{1}{2}$ and $x = \frac{1}{3}$, and guessed that

$\sum_{k=1}^{+\infty} x^k$ converges in those cases; i.e. that

$\frac{1}{2} + \frac{1}{2^2} + \dots$ and $\frac{1}{3} + \frac{1}{3^2} + \dots$ make sense

as numbers. Here, we prove:

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Theorem: • $\sum_{k=1}^{+\infty} x^k$ converges when $|x| < 1$

(in which case $\sum_{k=1}^{+\infty} x^k = \frac{x}{1-x}$).

• $\sum_{k=1}^{+\infty} x^k$ diverges when $|x| \geq 1$.

In particular,

- $\sum_{k=1}^{+\infty} x^k = +\infty$ for $x \geq 1$

- $\sum_{k=1}^{+\infty} x^k$ doesn't make sense
for $x \leq -1$.

Proof: $s_n = x^1 + x^2 + \dots + x^n$. We need to find the limiting behaviour of s_n as $n \rightarrow +\infty$:

We notice that $x \cdot s_n = x \cdot (x^1 + x^2 + \dots + x^n)$

$$= x^2 + x^3 + \dots + x^{n+1} = \\ = s_n - x + x^{n+1}.$$

$$\text{So, } x \cdot s_n = s_n - x + x^{n+1}$$

$$\rightarrow s_n(1-x) = x(1-x^n)$$

$$\Rightarrow \boxed{s_n = \frac{x}{1-x} \cdot (1-x^n)} \text{ for } x \neq 1$$

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Thus:

- For $|x| < 1$, $s_n \underset{n \rightarrow \infty}{\longrightarrow} \frac{x}{1-x} \cdot (1-0) = \frac{x}{1-x}$

(because $x^n \underset{n \rightarrow \infty}{\longrightarrow} 0$ in this case) $\sum_{k=1}^{+\infty} x^k = \frac{x}{1-x}$

- For $x > 1$, $s_n \underset{n \rightarrow \infty}{\longrightarrow} \frac{x}{1-x} (1-(+\infty)) = +\infty$

(because $x^n \underset{n \rightarrow \infty}{\longrightarrow} +\infty$ in this case).

So, $\sum_{k=1}^{+\infty} x^k = +\infty$ (the series diverges, but the infinite sum makes sense: it equals $+\infty$)

- For $x = 1$, we showed earlier that $s_n \underset{n \rightarrow \infty}{\longrightarrow} +\infty$ as well. So, $\sum_{k=1}^{+\infty} x^k = +\infty$.
- For $x \leq -1$, $\lim_{n \rightarrow \infty} x^n$ doesn't exist

in $\text{RV}\{+\infty\}$, so $\lim_{n \rightarrow \infty} s_n$ doesn't

exist either. So, $\sum_{k=1}^{+\infty} x^k$ is divergent,

and the infinite sum doesn't make sense.



Exercise: • Find $\sum_{k=1}^{+\infty} x^k$ for $x = \frac{1}{2}, x = \frac{1}{3}, x = -\frac{1}{2}$

- Show that $\sum_{k=1}^{+\infty} x^k < 0$ when $-1 < x < 0$. Does that make sense intuitively?

$$x = -\frac{1}{4}$$

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(Hint: Show that the partial sums are ≤ 0 .)

We now prove a very important theorem, which at the heart of the meaning of series convergence:

Theorem: If $\sum_{k=1}^{+\infty} a_k$ converges, then

$$\lim_{k \rightarrow +\infty} a_k = 0.$$

Idea: Let us see this first intuitively.

$\sum_{k=1}^{+\infty} a_k$ converges, which means that

the partial sums s_n of $\sum_{k=1}^{+\infty} a_k$

converge to some number s ; in other words, for large n , the s_n 's "cluster" around s :

So, for large n , the s_n 's don't differ too much from each other; they are all "pretty much" equal to s .

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That is, s_n and s_{n+1} are pretty much the same for large n . But:

$$s_n = a_1 + \dots + a_n,$$

and $s_{n+1} = a_1 + \dots + a_n + a_{n+1}.$

So, ~~s_{n+1}~~ and s_n differ by a_{n+1} ;

therefore a_{n+1} is pretty much 0 for large n .

Proof: $\sum_{k=1}^{+\infty} a_k$ converges. So, $\exists s \in \mathbb{R}$

$$\text{s.t. } \lim_{n \rightarrow +\infty} s_n = s.$$

$$\text{So, } \lim_{n \rightarrow +\infty} s_{n+1} = s \text{ as well}$$

(because $(s_{n+1})_{n \in \mathbb{N}}$ is the sequence

(s_2, s_3, s_4, \dots) , which ~~has~~ has the same

limit as $(s_1, s_2, s_3, \dots) = (s_n)_{n \in \mathbb{N}}$, because

limits of sequences don't depend on their first term).

$$\text{Thus, } \lim_{n \rightarrow +\infty} (s_{n+1} - s_n) = \lim_{n \rightarrow +\infty} s_{n+1} - \lim_{n \rightarrow +\infty} s_n = s - s = 0.$$

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But $s_{n+1} - s_n = a_{n+1}$. So,

$\lim_{n \rightarrow \infty} a_{n+1} = 0$, which means that

$\lim_{n \rightarrow \infty} a_n = 0$ (because, as we just mentioned, the limit of a sequence doesn't depend on the first term of the sequence) 

from this Theorem we get our first, most basic test of series convergence!

Here it is:



Theorem (preliminary test): If $a_k \xrightarrow[k \rightarrow \infty]{} 0$,

then $\sum_{k=1}^{+\infty} a_k$ diverges.

Proof: Suppose that $\sum_{k=1}^{+\infty} a_k$ converges. Then,

by the previous theorem, $a_k \xrightarrow[k \rightarrow \infty]{} 0$. This

is a contradiction, so $\sum_{k=1}^{+\infty} a_k$ diverges. 

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ex: • $\sum_{k=1}^{+\infty} x^k$ diverges when $|x| > 1$ (second proof)

for $|x| > 1$, $(x^k)_{k \in \mathbb{N}}$ doesn't converge to 0.

So, $\sum_{k=1}^{+\infty} x^k$ diverges.



The converse of the preliminary test

is NOT NECESSARILY TRUE. I.e., if

$a_k \xrightarrow[k \rightarrow +\infty]{} 0$, it isn't necessarily true

that $\sum_{k=1}^{+\infty} a_k$ converges. It is true

sometimes (e.g. $\sum_{k=1}^{+\infty} x^k$ for $|x| < 1$),

but false others (e.g. $\sum_{k=1}^{+\infty} \frac{1}{k}$: the

harmonic series).

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Lecture 2

26 Aug 2016.

We have shown that

If $a_k \rightarrow 0$ as $k \rightarrow \infty$, then $\sum_{k=1}^{\infty} a_k$ diverges.

This is the first, preliminary test for series convergence; one checks whether $a_k \rightarrow 0$

or not. However, the converse is not true: if $a_k \rightarrow 0$ as $k \rightarrow \infty$,

test is inconclusive, and we need to check more. In other words:

If $a_k \rightarrow 0$ as $k \rightarrow \infty$, it isn't necessarily true

that $\sum_{k=1}^{\infty} a_k$. It is true sometimes,

for example for $\sum_{k=1}^{\infty} x^k$ when $|x| < 1$,

but false others, for example:

Harmonic Series: $\sum_{k=1}^{\infty} \frac{1}{k}$. This

series diverges; in fact, $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

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But of course $\frac{1}{k} \xrightarrow[k \rightarrow \infty]{} 0$.

Proof of divergence of harmonic series:

We look at the partial sums with 2^n terms:

$$S_{2^1} = 1 + \frac{1}{2}$$

$$\begin{aligned} S_{2^2} = S_4 &= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}} \geq 1 + \frac{1}{2} + \frac{1}{2} \\ &\geq \frac{1}{2} + \frac{1}{2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} S_{2^3} &= S_4 + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}} \geq S_4 + \frac{1}{2} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\ &\vdots \end{aligned}$$

It is clear that, by induction,

$$S_{2^n} \geq 1 + \frac{n}{2} \xrightarrow[n \rightarrow \infty]{} +\infty$$

Now, if the series converged, there would exist some $s \in \mathbb{R}$ s.t. $S_n \xrightarrow[n \rightarrow \infty]{} s$.

And therefore $(S_{2^n})_{n \in \mathbb{N}}$ would also converge

to s , as a subsequence of a convergent sequence (intuitively, if the ~~terms~~ s_n 's

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cluster around s for large n , then so do the s_n 's). So, $\sum_{k=1}^{+\infty} \frac{1}{k}$ doesn't converge, i.e. it diverges.

AThis proof doesn't show that $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$;

it just shows that $\sum_{k=1}^{+\infty} \frac{1}{k}$ diverges. If

you are curious, however, $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$ is immedi-

ate because $\frac{1}{k} \geq 0 \forall k$ and  some

subsequence of the partial sums tends to $+\infty$. Indeed, since $\frac{1}{k} \geq 0 \forall k \in \mathbb{N}$, the

partial sums form an increasing sequence

$(s_n)_{n \in \mathbb{N}}$. And it is a theorem that an

increasing sequence has a limit in $\mathbb{R} \cup \{+\infty\}$

(its limit always exists). So, either

$s_n \xrightarrow[n \rightarrow \infty]{} s \in \mathbb{R}$ or $s_n \xrightarrow[n \rightarrow \infty]{} +\infty$. Since we have

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excluded the first, the second must hold; so $s_n \xrightarrow[n \rightarrow \infty]{} +\infty$, so $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$.

We will now look at some more tests for convergence. In principle, these tests work for series with non-negative terms. So, to apply them to any series, we have to replace its terms with their absolute values, and check convergence for the new series; if this new series converges, then the original one does too. More precisely:

→ Def.: (i) We say that $\sum_{k=1}^{+\infty} a_k$ converges

absolutely if $\sum_{k=1}^{+\infty} |a_k|$ converges.

(ii) If $\sum_{k=1}^{+\infty} a_k$ converges, but doesn't

converge absolutely (i.e. $\sum_{k=1}^{+\infty} |a_k| = +\infty$),

then we say that $\sum_{k=1}^{+\infty} a_k$ converges

conditionally.

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Theorem: If $\sum_{k=1}^{+\infty} a_k$ converges absolutely,

then it converges

(i.e., $\sum_{k=1}^{+\infty} |a_k|$ converges $\Rightarrow \sum_{k=1}^{+\infty} a_k$ converges)

It is this theorem that makes it worth checking for absolute convergence! It says that absolute convergence is a stronger type of convergence.

ex: • $\sum_{k=1}^{+\infty} \frac{(-1)^k}{2^k}$ converges absolutely,

because $\sum_{k=1}^{+\infty} \left| \frac{(-1)^k}{2^k} \right| = \sum_{k=1}^{+\infty} \frac{1}{2^k}$ converges

So, $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$ converges.

• $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$ doesn't converge absolutely,

because $\sum_{k=1}^{+\infty} \left| \frac{(-1)^k}{k} \right| = \sum_{k=1}^{+\infty} \frac{1}{k}$ diverges.

However, we will see later that

$\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$ converges; thus, it converges conditionally.

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Let us now see some tests for convergence

of a series $\sum_{k=1}^{+\infty} |a_k|$, that have to do with

comparing the series with a quantity we understand.

→ Comparison tests:

I Comparing with a series $\sum_{k=1}^{+\infty} b_k$, $b_k \geq 0 \forall k \in \mathbb{N}$:

I.1 Basic comparison test:

- If $|a_k| \leq b_k \forall k \in \mathbb{N}$, then:

if $\sum_{k=1}^{+\infty} b_k$ converges, then $\sum_{k=1}^{+\infty} |a_k|$ converges

- If $|a_k| \geq b_k \forall k \in \mathbb{N}$, then:

if $\sum_{k=1}^{+\infty} b_k = +\infty$, then $\sum_{k=1}^{+\infty} |a_k| = +\infty$

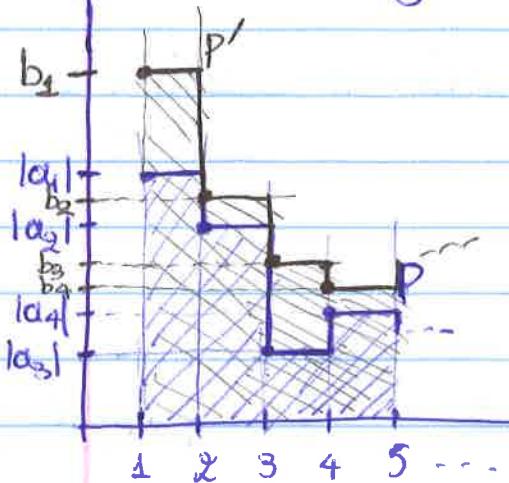
To visualise this, it helps to bear in mind that a series is actually the integral of a piecewise constant function. More precisely:

$|a_k|$ is the area of a parallelogram with sides

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of lengths 1 and $|a_k|$. So, consider

the following function: $a(x) = \begin{cases} 0, & x < 1 \\ |a_k|, & x \in [k, k+1], \\ & \forall k \in \mathbb{N} \end{cases}$



Then, the area under P equals the sum of sizes of parallelograms: $|a_1| + |a_2| + \dots = \sum_{k=1}^{+\infty} |a_k|$.

Now, we also consider

$$b(x) = \begin{cases} 0, & x < 1 \\ b_k, & x \in [k, k+1], \\ & \forall k \in \mathbb{N} \end{cases}$$

Similarly,

the area under P' equals the sum of sizes of some other parallelograms: $b_1 + b_2 + \dots = \sum_{k=1}^{+\infty} b_k$.

So, if P' is above P (i.e. $|a_k| \leq b_k$

$\forall k \in \mathbb{N}$), then the area under P is

at most the area under P' , i.e.

$$\sum_{k=1}^{+\infty} |a_k| \leq \sum_{k=1}^{+\infty} b_k ; \text{ so, if } \sum_{k=1}^{+\infty} b_k \text{ converges.}$$

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(the sum is finite) then $\sum_{k=1}^{+\infty} |a_k|$ converges

(i.e. this area has to be finite too).

On the other hand, if P' is below P
(i.e. $b_k \leq |a_k| \forall k \in \mathbb{N}$), then the area

under P is larger than the area under P'

That is, $\sum_{k=1}^{+\infty} |a_k| \geq \sum_{k=1}^{+\infty} b_k$, so, if

the area $\sum_{k=1}^{+\infty} b_k$ is infinite, then the area

$\sum_{k=1}^{+\infty} |a_k|$ is infinite too.



This is not a proof; it is as obscure as saying simply that if $|a_k| \leq b_k \forall k \in \mathbb{N}$, then $\sum_{k=1}^{+\infty} |a_k| \leq \sum_{k=1}^{+\infty} b_k$,

so, if $\sum_{k=1}^{+\infty} b_k$ finite, then $\sum_{k=1}^{+\infty} |a_k|$ finite. This

is obscure because it doesn't explain that the partial sums of $\sum_{k=1}^{+\infty} |a_k|$ converge.

The official proof uses that $\sum_{k=1}^{+\infty} |a_k|$ has

partial sums that form an increasing sequence.

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$(s_n)_{n \in \mathbb{N}}$. So (see earlier remark),

either $s_n \xrightarrow[n \rightarrow \infty]{\text{---}} +\infty$ or $s_n \xrightarrow[n \rightarrow \infty]{\text{---}} s$, for

some $s \in \mathbb{R}$. If $|a_k| \geq b_k \ \forall k \in \mathbb{N}$,

then $s_n \geq$ the n -th partial sum of $\sum_{k=1}^{+\infty} b_k$

$\forall n$, so, if $\sum_{k=1}^{+\infty} b_k = +\infty$ (in which

case $\sum_{n=1}^{\infty} s_n = +\infty$), we also have $s_n \xrightarrow[n \rightarrow \infty]{\text{---}} +\infty$,

so $\sum_{k=1}^{+\infty} |a_k| = +\infty$. On the other hand,

if $|a_k| \leq b_k \ \forall k \in \mathbb{N}$, then $s_n \leq \sum_{n=1}^{\infty} s_n$ $\forall n \in \mathbb{N}$,

so, since $\lim_{n \rightarrow \infty} s_n$ exists in \mathbb{R} ,

we have that $(s_n)_{n \in \mathbb{N}}$ is bounded from

above by some $M \in \mathbb{R}$. So, ~~$s_n \xrightarrow[n \rightarrow \infty]{\text{---}} +\infty$~~

$s_n \leq M \ \forall n \in \mathbb{N}$, so it is impossible to

have $s_n \xrightarrow[n \rightarrow \infty]{\text{---}} +\infty$. So, $s_n \xrightarrow[n \rightarrow \infty]{\text{---}} s$, 
 i.e. $\sum_{k=1}^{+\infty} |a_k|$ converges.

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The point is that, in mathematics, a convergent series is nothing more than a (well-defined) integral. So, this is a good perspective, especially given that we will use the same approach when we compare series with integrals later.

ex: • $\sum_{k=2}^{+\infty} \frac{1}{\ln k}$: does it converge or diverge?

Answer: It ~~diverges~~ converges: $\ln k \leq k \forall k \geq 2$,

so $\frac{1}{\ln k} \geq \frac{1}{k} \quad \forall k \geq 2$. So,

since $\sum_{k=2}^{+\infty} \frac{1}{k} = +\infty$, $\sum_{k=2}^{+\infty} \frac{1}{\ln k} = +\infty$ as well.

• $\sum_{k=1}^{+\infty} \frac{\cos(kx)}{2^k}$: does it converge or diverge?

Answer: It converges: $|\cos(kx)| \leq 1 \quad \forall k \in \mathbb{N}$,

so $\left| \frac{\cos(kx)}{2^k} \right| \leq \frac{1}{2^k} \quad \forall k \in \mathbb{N}$, and

$\sum_{k=1}^{+\infty} \frac{1}{2^k}$ converges. So, $\sum_{k=1}^{+\infty} \left| \frac{\cos(kx)}{2^k} \right|$ converges,

so $\sum_{k=1}^{+\infty} \frac{\cos(kx)}{2^k}$ converges.

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I2) Limiting comparison test: (remember, we still have $b_k \geq 0 \forall k \in \mathbb{N}$)

Suppose that $b_k \neq 0 \forall k \in \mathbb{N}$.

I2i) If $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k}$ exists in \mathbb{R} and $\sum_{k=1}^{+\infty} b_k$ converges,
then $\sum_{k=1}^{+\infty} |a_k|$ converges.

| Proof if you are curious: $\left(\frac{|a_k|}{b_k} \right)_{k \in \mathbb{N}}$

converges, so $\exists M \in \mathbb{R}$ s.t. $\frac{|a_k|}{b_k} \leq M \forall k \in \mathbb{N}$,

$$\text{i.e. } |a_k| \leq M \cdot b_k \forall k \in \mathbb{N}.$$

$$\text{Since } \sum_{k=1}^{+\infty} M \cdot b_k = M \cdot \sum_{k=1}^{+\infty} b_k$$

the $\frac{|a_k|}{b_k}$ are close to $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k}$ for large k .
to $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k}$ for test

converges, by the comparison test $\sum_{k=1}^{+\infty} |a_k|$ also converges.

I2ii) If $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k}$ exists in $(0, +\infty]$ and

$$\sum_{k=1}^{+\infty} b_k = +\infty, \text{ then } \sum_{k=1}^{+\infty} |a_k| = +\infty.$$

| Proof if you are curious: Since $\lim_{k \rightarrow \infty} \frac{|a_k|}{b_k} \in (0, +\infty]$,

there exists $M > 0$ s.t. $M \leq \frac{|a_k|}{b_k}$ for large k .

So, $|a_k| \geq M \cdot b_k$ for large k . And since

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$$\sum_{k=1}^{+\infty} M \cdot b_k = M \cdot \sum_{k=1}^{+\infty} b_k = +\infty, \text{ by}$$

the comparison test we also have

$$\sum_{k=1}^{+\infty} |a_k| = +\infty.$$

ex: $\sum_{k=1}^{+\infty} \frac{k+1}{k^2+2}$. We guess that this

series should have the same convergence behaviour as $\sum_{k=1}^{+\infty} \frac{1}{k}$, as $\frac{k+1}{k^2+2}$

should behave like $\frac{1}{k}$ for large k .

Indeed:

$$\frac{\frac{k+1}{k^2+2}}{\frac{1}{k}} = \frac{k(k+1)}{k^2+2} = \frac{k^2+k}{k^2+2} \xrightarrow[k \rightarrow +\infty]{(0,+\infty)} 1.$$

So, by the limiting comparison test,

since $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$, $\sum_{k=1}^{+\infty} \frac{k+1}{k^2+2} = +\infty$ as well.

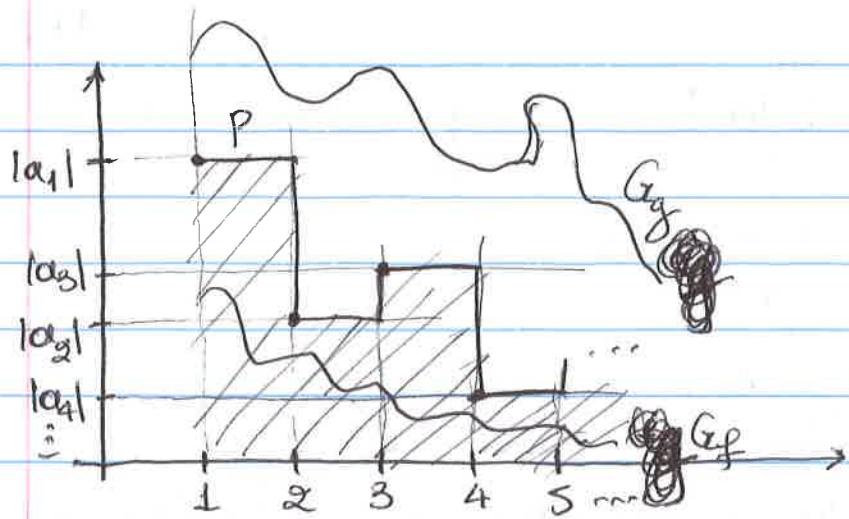
II Comparing with an integral:

Some general discussion:

We once again consider

$$a(x) = \begin{cases} 0, & x < 1 \\ |a_k|, & x \in [k, k+1], \forall k \in \mathbb{N}. \end{cases}$$

(B).



- Suppose I can find $f: (1, +\infty) \rightarrow \mathbb{R}_{\geq 0}$,

whose graph is below P (maybe intersecting it),
so that $\int_1^{+\infty} f(x) dx = +\infty$.

$\underbrace{\hspace{1cm}}_{\text{area under } G_f}$

Then, $\sum_{k=1}^{+\infty} |a_k| \geq \int_1^{+\infty} f(x) dx = +\infty$.

$\underbrace{\hspace{1cm}}_{\text{area under } P}$

- Suppose I can find $g: (1, +\infty) \rightarrow \mathbb{R}_{\geq 0}$,

whose graph is above P (maybe intersecting it),

so that $\int_1^{+\infty} g(x) dx < +\infty$.

$\underbrace{\hspace{1cm}}_{\text{area under } G_g}$

(14).

Then, $\sum_{k=1}^{+\infty} |a_k| \leq \int_1^{+\infty} g(x) dx < +\infty$,

area under

so $\sum_{k=1}^{+\infty} |a_k|^p$ converges.

This idea is usually in literature as the following integral test:

- Suppose that $|a_1| \geq |a_2| \geq |a_3| \geq \dots$

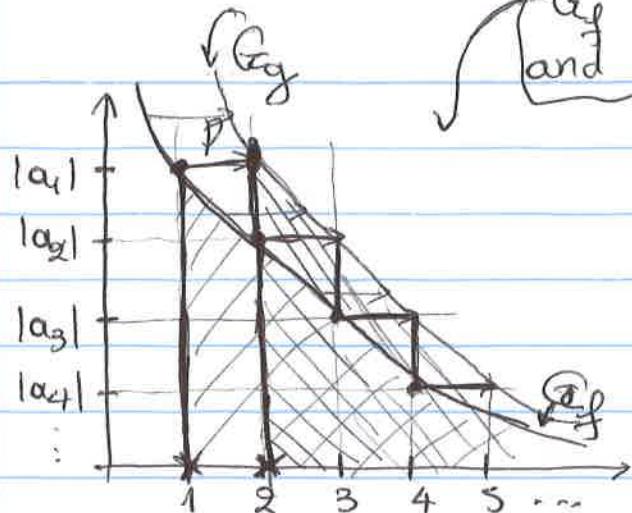
Take the formulae for $|a_k|$, and extend it to a function $f: (1, +\infty) \rightarrow \mathbb{R}_{\geq 0}$.

e.g.: If $a_k = \frac{1}{k}$ $\forall k \in \mathbb{N}$,

$$\text{consider } f(x) = \frac{1}{x} \quad \forall x > 1$$

- Suppose that f non-increasing
(this is not in the Boas book,
but it is vital).

Then, we have the following picture:



G_f is under P on (1, +∞),
and G_g is above P on (2, +∞)

(The area under P)
 $= \sum_{k=1}^{+\infty} |a_k|$ ~~is~~

\geq (the area under)
 G_f
 $= \int_1^{+\infty} f(x) dx.$

So, if $\int_1^{+\infty} f(x) dx = +\infty,$
 then $\sum_{k=1}^{+\infty} |a_k| = +\infty.$

Translate the graph of f by 1 horizontally.
 This way, (the area under P from
 x=2 onwards)

$$= \sum_{k=2}^{+\infty} |a_k| \leq (\text{the area under } G_g \text{ from } x=2 \text{ onwards}) =$$

$$= \int_2^{+\infty} g(x) dx = \int_2^{+\infty} f(x-1) dx = \int_1^{+\infty} f(x) dx$$

So, if $\int_1^{+\infty} f(x) dx < +\infty,$
 then $\sum_{k=2}^{+\infty} |a_k| < +\infty.$

To sum up, the integral test (the one most commonly used) says the following:

Let $\sum_{k=1}^{+\infty} |a_k|$ be a series.

If the terms of the series are non-increasing (i.e. $|a_k| \geq |a_{k+1}| \forall k \in \mathbb{N}$) then it makes sense to attempt the integral test.

In that case, we extend the formula for a_k to all $x \in (1, +\infty)$. This way we get $f: (1, +\infty) \rightarrow \mathbb{R}$. If f is non-increasing (i.e. $f(x) \geq f(y) \forall x \leq y$),

then the integral test will work.

In particular:

- If $\int_1^{+\infty} f(x) dx = +\infty \Rightarrow \sum_{k=1}^{+\infty} |a_k| = +\infty$.

- If $\int_1^{+\infty} f(x) dx < +\infty \Rightarrow \sum_{k=1}^{+\infty} |a_k|$ converges.

(2)

Applications of the integral test:

$$\rightarrow \sum_{k=1}^{+\infty} \frac{1}{k} = +\infty \quad (\text{a second proof}):$$

We have that the terms of this series are non-negative, and non-increasing ($\frac{1}{k} \geq \frac{1}{k+1}$ $\forall k \in \mathbb{N}$). So, we can attempt the integral test.

We extend the formula for $\frac{1}{k}$ to all $x \in (1, +\infty)$; we thus get $f(x) = \frac{1}{x}$, $\forall x \in (1, +\infty)$.

Since f is non-increasing ($\frac{1}{x} \geq \frac{1}{y}$ $\forall x \leq y$),

the integral test will work. We have

$$\int_1^{+\infty} f(x) dx = \int_1^{+\infty} \frac{1}{x} dx = [\ln x]_1^{+\infty} = \ln(+\infty) - \ln 1 = +\infty,$$

so $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$.

$$\rightarrow \boxed{\text{p-series : } \sum_{k=1}^{+\infty} \frac{1}{k^p}, p > 0}$$

We will show that $\sum_{k=1}^{+\infty} \frac{1}{k^p}$ converges for $p > 1$

and $\sum_{k=1}^{+\infty} \frac{1}{k^p} = +\infty$ for $p \leq 1$



for $p=1$, the p-series becomes the harmonic series.

(3)

Proof: Fix $p \in \mathbb{R}$ ($p \neq 1$, because we have done that already).

The terms of $\sum_{k=1}^{+\infty} \frac{1}{k^p}$ are non-negative,

and non-increasing $\left(\frac{1}{k^p} \geq \frac{1}{(k+1)^p} \quad \forall k \in \mathbb{N}, \text{ as } p > 0 \right)$,

so it makes sense to attempt the integral test.

We extend the formula for $\frac{1}{k^p}$ to all

$x \in (1, +\infty)$; we thus get $f(x) = \frac{1}{x^p}$, $\forall x \in (1, +\infty)$.

Since f is non-increasing $\left(\frac{1}{x^p} \geq \frac{1}{y^p} \quad \forall x \leq y \right)$
 $\left(\text{in } (1, +\infty), \text{ as } p > 0 \right)$

the integral test will work. We have:

$$\int_1^{+\infty} f(x) dx = \int_1^{+\infty} \frac{1}{x^p} dx \stackrel{p \neq 1}{=} \left[\frac{x^{-p+1}}{-p+1} \right]_1^{+\infty} = \frac{1}{1-p} \cdot \left[\frac{1}{x^{p-1}} \right]_1^{+\infty}.$$

(4)

- When $p-1 > 0 \Leftrightarrow p > 1$, we have $\frac{1}{x^{p-1}} \xrightarrow[x \rightarrow +\infty]{} 0$, so

$$\frac{1}{1-p} \cdot \left[\frac{1}{x^{p-1}} \right]_1^{+\infty} = \frac{1}{1-p} (0 - 1) = \frac{-1}{1-p} = \frac{1}{p-1} < \infty,$$

i.e. $\int_1^{+\infty} f(x) dx < \infty$, so $\sum_{k=1}^{+\infty} \frac{1}{k^p}$ converges.

- When $p-1 < 0 \Leftrightarrow p < 1$, we have $\frac{1}{x^{p-1}} = x^{-(p-1)} \xrightarrow[x \rightarrow +\infty]{} +\infty$,

$$\text{so } \frac{1}{1-p} \cdot \left[\frac{1}{x^{p-1}} \right]_1^{+\infty} = \frac{1}{1-p} \cdot (+\infty - 1) = +\infty,$$

i.e. $\int_1^{+\infty} f(x) dx = +\infty$, so $\sum_{k=1}^{+\infty} \frac{1}{k^p} = +\infty$.

A So far, we have only seen convergence tests for series with non-negative terms; for series with also negative terms, we have only seen that they converge if they converge absolutely. The following tests cover series with negative terms too!

→ **The ratio test:** Let $\sum_{k=1}^{+\infty} a_k$ be a series

with $a_k \neq 0$, $k \in \mathbb{N}$.

Suppose that the $\lim_{k \rightarrow +\infty} \frac{|a_{k+1}|}{|a_k|} = l$ exists (in $\mathbb{R} \cup \{\infty\}$)

if not, the test is inconclusive.

(5)

- If $\rho < 1$, then $\sum_{k=1}^{+\infty} |\alpha_k|$ converges.

- If $\rho > 1$, then $\sum_{k=1}^{+\infty} \alpha_k$ diverges.

- If $\rho = 1$, then the test is inconclusive, and we need to check more.

ex: • $\sum_{k=1}^{+\infty} \left(\frac{1}{k}\right)^{\rho}$. $\frac{1}{k} \neq 0 \forall k \in \mathbb{N}$. And

$$\frac{|\alpha_{k+1}|}{|\alpha_k|} = \frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{k}{k+1} \xrightarrow[k \rightarrow +\infty]{} 1, \text{ so the ratio test is inconclusive.}$$

test is inconclusive (but we know that $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$).

- $\sum_{k=1}^{+\infty} \left(\frac{1}{k^2}\right)^{\rho}$. $\frac{1}{k^2} \neq 0 \forall k \in \mathbb{N}$. And

$$\frac{|\alpha_{k+1}|}{|\alpha_k|} = \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \frac{k^2}{(k+1)^2} = \frac{k^2}{k^2 + 2k + 1} \xrightarrow[k \rightarrow +\infty]{} 1,$$

so the ratio test is inconclusive (but we know

that $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges: it is the p-series for $p=2$).

⑥

• $\sum_{k=1}^{+\infty} \frac{x^k}{k^p} = \infty$, for $p > 0$:

$$\frac{|x_{k+1}|}{|x_k|} = \frac{|x^{k+1}|}{\frac{|x^k|}{k^p}} = \frac{|x|^{k+1} k^p}{(k+1)^p |x|^k} = |x| \left(\frac{k}{k+1}\right)^p \xrightarrow[k \rightarrow \infty]{} |x|.$$

So:

- If $|x| \leq 1$, then $\sum_{k=1}^{+\infty} \frac{x^k}{k^p} < \infty$ (independently of p)
- If $|x| > 1$, then $\sum_{k=1}^{+\infty} \frac{x^k}{k^p} = \infty$ (independently of p).
- If $|x|=1$, the ratio test is inconclusive. However,

for $x=1$, the series becomes the p-series,
whose convergence behaviour we
have studied! And,

for $x=-1$, the series is alternating, and satisfies
the (simple test) of convergence for
alternating series; so it converges

see later
section.

 It is easy to see that $\frac{x^k}{k^p} \xrightarrow[k \rightarrow \infty]{} 0$ for $|x| < 1$.

However, we have:

$$\frac{x^k}{k^p} \xrightarrow[k \rightarrow \infty]{} \infty \text{ for } x > 1$$

(ex: $\frac{2^k}{k^3} \xrightarrow[k \rightarrow \infty]{} \infty$, $\frac{3^k}{k^{10}} \xrightarrow[k \rightarrow \infty]{} \infty$, etc).

(problem
2.4 p.5 of
book)

(7)

You may find the following test more useful, sometimes, than the ratio test; for instance, if $a_k = 0$ for infinitely many k 's:

not in
the book.

→ **The root test:** Let $\sum_{k=1}^{+\infty} a_k$ be a series.

Suppose that $\lim_{k \rightarrow +\infty} \sqrt[k]{|a_k|}$ exists (in $\mathbb{R} \cup \{+\infty\}$).

if not, the test is inconclusive.

- If $\ell < 1$, then $\sum_{k=1}^{+\infty} |a_k|$ converges.

- If $\ell > 1$, then $\sum_{k=1}^{+\infty} |a_k|$ converges because the root test for $\sum_{k=1}^{+\infty} |a_k|$ is the same as for $\sum_{k=1}^{+\infty} a_k$.

- If $\ell = 1$, then the test is inconclusive, and we need to check more.

ex: • $\sum_{k=1}^{+\infty} \frac{1}{k}$: We have

$$\sqrt[k]{k} \xrightarrow[k \rightarrow +\infty]{} 1 \quad \begin{cases} \text{see ex.3} \\ \text{in p. 5} \\ \text{of book} \end{cases}$$

so $\sqrt[k]{\frac{1}{k}} \xrightarrow[k \rightarrow +\infty]{} 1$, so the root test is

inconclusive (but we know that $\sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$).

(8)

- $\sum_{k=1}^{\infty} \frac{1}{k^2}$: We have $\sqrt[k]{\frac{1}{k^2}} = \frac{1}{(\sqrt[k]{k})^2} \xrightarrow{k \rightarrow \infty} \frac{1}{1^2} = 1$,

so the test is inconclusive (but we know that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges).

- $\sum_{k=1}^{\infty} \left(\frac{1}{2} + \frac{1}{k} \right)^k$: We have $\sqrt[k]{\left(\frac{1}{2} + \frac{1}{k} \right)^k} = \frac{1}{2} + \frac{1}{k} \xrightarrow{k \rightarrow \infty} \frac{1}{2}$.

Since $\frac{1}{2} < 1$, the series converges.



A common situation where we apply the root test is when a_k involves $\left(1 + \frac{1}{k}\right)^k$.

It holds that

$$\boxed{\left(1 + \frac{1}{k}\right)^k \xrightarrow{k \rightarrow \infty} e}.$$

You will need this in exercises. You can prove it using ex. 2, 3 in Section 2 of p. 5 of the book (i.e., ideas from the examples, rather than the examples themselves).

Another important thing to remember here is that

$$\boxed{\sqrt[k]{a^k} \xrightarrow{k \rightarrow \infty} 1, \text{ if } a > 0.}$$

(again, you can prove this using ideas from ex. 3, p. 5 of the book).

(9)



Note that the comparison tests cannot verify whether a series with both positive and negative terms diverges; but the ratio and root tests can!

In general, when we have a series with both positive and negative terms, to test it for convergence we can test it for absolute convergence, or use the ratio or root tests. There is however the very special case of alternating series, where the situation is simple :

Alternating series: This is defined as a series

whose terms have signs alternating from + to -.

I.e., an alternating series is any series of the form $\sum_{k=1}^{+\infty} (-1)^k \cdot a_k$ or $\sum_{k=1}^{+\infty} (-1)^{k+1} \cdot d_k$

where $a_k \geq 0 \quad \forall k \in \mathbb{N}$

There is a very simple test for alternating series:

Alternating series test:

Let $\sum_{k=1}^{+\infty} a_k$ be an alternating series. If

$$\begin{cases} a_k \xrightarrow{k \rightarrow \infty} 0 \\ \text{and } |a_k| \text{ non-increasing (i.e. } |a_k| \geq |a_{k+1}| \text{ for all } k \in \mathbb{N}) \end{cases}$$

(10)

then $\sum_{k=1}^{+\infty} a_k$ converges.

ex: The series $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$, $\sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt{k}}$, $\sum_{k=2}^{+\infty} \frac{(-1)^k}{\ln k}$ all converge, because their terms tend to 0 and they are non-increasing in absolute value.



Note that none of the three series above converges absolutely! So, we have proved that there exist series that only converge conditionally. I.e.:

not absolute convergence



not convergence.

Conditionally convergent series:

If a series is conditionally convergent, it means that it converges, but doesn't converge absolutely.

Let $\sum_{k=1}^{+\infty} a_k$ be such a series.

In this case, we always have :

$$\left. \begin{array}{l} \text{sum of positive terms} = +\infty \\ \text{and sum of negative terms} = -\infty \\ \text{Also: } a_k \xrightarrow[k \rightarrow +\infty]{} 0 \quad (\text{or } \sum_{k=1}^{+\infty} a_k \text{ would diverge!}) \end{array} \right\}$$

(11)

Thanks to these 3 facts, we can rearrange the terms of the series so that the new series converges to any number we want! Indeed, pick your favourite number;

say, $\frac{1}{3}$. Pick as many positive terms (as you

need to add up to at least $\frac{1}{3}$) ; you can

do this, because the sum of positive terms is $+\infty$.

Now, add as many negative terms (as you need to go below $\frac{1}{3}$) ; you can do that again because

the sum of negative terms is $-\infty$. Then, add

the largest possible remaining positive terms until you go above $\frac{1}{3}$; then add the smallest

possible negative terms until you go below $\frac{1}{3}$), etc.

Since $a_k \xrightarrow[k \rightarrow \infty]{} 0$, in the end we will just be oscillating

around $\frac{1}{3}$, but very close to it; so the

total sum will be $\frac{1}{3}$ in the limit.

(19)

On the other hand, if we sum the positive

terms faster than the negative, we can make
the sum too;

and if we sum the negative terms faster
than the positive, we can make the sum $-\infty$

So: here we see how much ORDER MATTERS

when we add infinitely many numbers!

A If a series has only positive or only negative
terms, then the order doesn't matter.

→ **Power series:**

→ Def: A power series is a series of the form

$$\sum_{k=1}^{+\infty} a_k x^k, \text{ for } x \in \mathbb{R},$$

or $\sum_{k=1}^{+\infty} a_k (x-a)^k, \text{ for } x \in \mathbb{R}, \text{ where}$
 $a \in \mathbb{R}$ is fixed.

We say that a is the center of the power series.

(13.)

Clearly, $\sum_{k=1}^{+\infty} a_k (x-a)^k$ converges for $x=a$ (as $\sum_{k=1}^{+\infty} a_k (a-a)^k = 0$)

What always holds for $\sum_{k=1}^{+\infty} a_k (x-a)^k$ is that

there exists some $R \geq 0$, s.t.

- $\sum_{k=1}^{+\infty} a_k (x-a)^k$ converges for $|x-a| < R$,
- $\sum_{k=1}^{+\infty} a_k (x-a)^k$ diverges for $|x-a| > R$, and
- $\sum_{k=1}^{+\infty} a_k (x-a)^k$ has a behaviour at $x=R$ and $x=-R$ that has to be checked in each of the two cases
(it can converge at both values,
or diverge at both values,
or converge at one and diverge at the other).

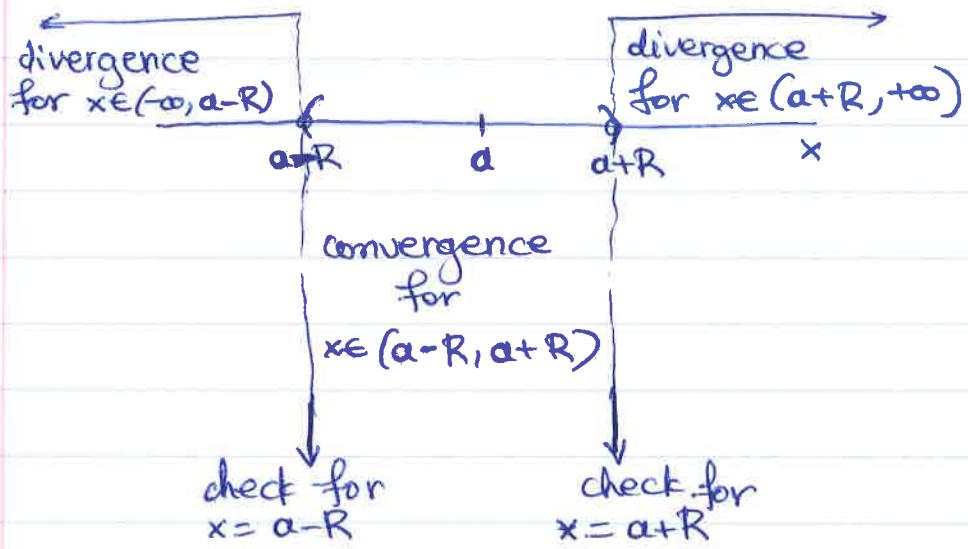
We call R the radius of convergence of

$\sum_{k=1}^{+\infty} a_k (x-a)^k$. In particular, the above means that:

- for $R=0$, $\sum_{k=1}^{+\infty} a_k (x-a)^k$ converges only for $x=a$.

(14).

- For $R > 0$, we have the following picture:



(12)

On the other hand, if we sum the positive terms faster than the negative, we can make the sum too;

and if we sum the negative terms faster than the positive, we can make the sum $-\infty$

So: here we see how much **ORDER MATTERS**

when we add infinitely many numbers!

A If a series has only positive or only negative terms, then the order doesn't matter.

→ **Power series:**

→ Def: A power series is a series of the form

$$\sum_{k=0}^{+\infty} a_k x^k, \text{ for } x \in \mathbb{R},$$

or $\sum_{k=0}^{+\infty} a_k (x-a)^k, \text{ for } x \in \mathbb{R}, \text{ where } a \in \mathbb{R} \text{ is fixed.}$

We say that a is the center of the power series

(13)

Clearly, $\sum_{k=0}^{+\infty} a_k (x-a)^k$ converges for $x=a$ (as $\sum_{k=0}^{+\infty} a_k (a-a)^k = 0$)

What always holds for $\sum_{k=0}^{+\infty} a_k (x-a)^k$ is that

there exists some $R \geq 0$, s.t.

- $\sum_{k=0}^{+\infty} a_k (x-a)^k$ converges for $|x-a| < R$,
- $\sum_{k=0}^{+\infty} a_k (x-a)^k$ diverges for $|x-a| > R$, and
- $\sum_{k=0}^{+\infty} a_k (x-a)^k$ has a behaviour at $x=R$ and $x=-R$ that has to be checked in each of the two cases
(it can converge at both values,
or diverge at both values,
or converge at one and diverge at the other).

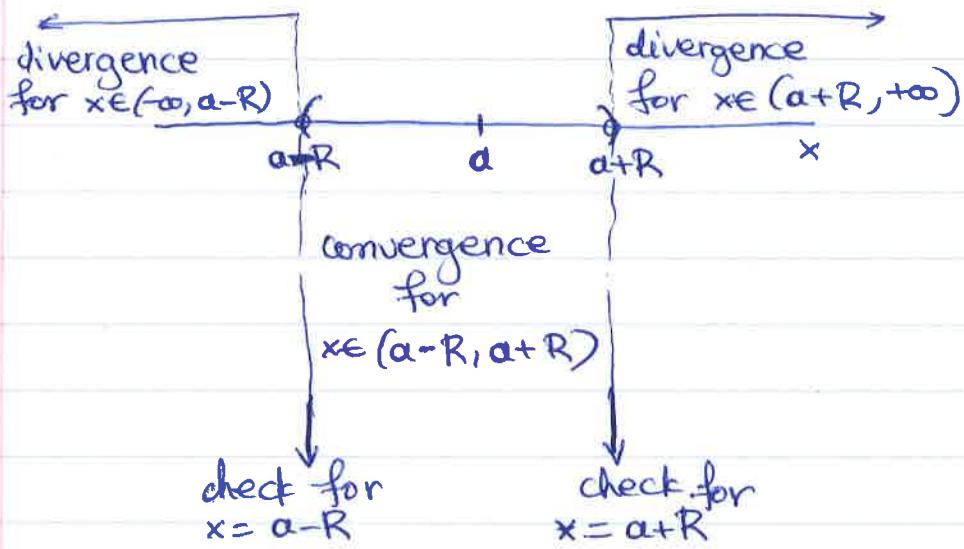
We call R the radius of convergence of

$\sum_{k=0}^{+\infty} a_k (x-a)^k$. In particular, the above means that:

- for $R=0$, $\sum_{k=0}^{+\infty} a_k (x-a)^k$ converges only for $x=a$.

(14)

- For $R > 0$, we have the following picture:



(1)

Lecture 4.

31 Aug 2016.

To prove that the above picture is true, one needs to know complex analysis. However, we will explain it via the ratio test, which we can use under the conditions \textcircled{k}_1 , \textcircled{k}_2 below:

Let $\sum_{k=0}^{+\infty} a_k x^k$ be a power series (we focus on center 0 for now),

with $a_k \neq 0 \quad \forall k \in \mathbb{N}$. Suppose that

$$\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|} = l$$

exists (in $\mathbb{R} \cup \{+\infty\}$)

(2)

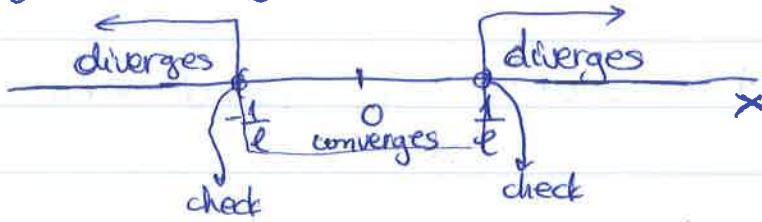
- If $\ell \cdot |x| < 1$, then $\sum_{k=0}^{\infty} \frac{a_k}{\ell^k} x^k$ converges absolutely.
- \Downarrow
 $|x| < \frac{1}{\ell}$

- If $\ell \cdot |x| > 1$, then $\sum_{k=0}^{\infty} \frac{a_k}{\ell^k} x^k$ diverges.
- \Downarrow
 $|x| > \frac{1}{\ell}$

- If $\ell \cdot |x| = 1$, then the test is inconclusive, and we have to test more.
- \Downarrow
 $|x| = \frac{1}{\ell}$
 \Downarrow
 $x = \frac{1}{\ell}$ or $x = -\frac{1}{\ell}$

The above means that the series $\sum_{k=0}^{\infty} \frac{a_k}{\ell^k} x^k$:

- converges for x inside $(-\frac{1}{\ell}, \frac{1}{\ell})$,
- diverges for $x \in (-\infty, -\frac{1}{\ell}) \cup (\frac{1}{\ell}, +\infty)$
- has behaviour that needs to be checked further for $x = -\frac{1}{\ell}$ and $x = \frac{1}{\ell}$.



(3)

So, its radius of convergence is $R = \frac{1}{l}$, where

$$l = \lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|}.$$

⚠ Check that $R = \infty$ for $l = 0$.

⚠ The same can be proved with the root test, in fact with one assumption fewer!
Try it!

→ To understand for which $x \in \mathbb{R}$ the general power series $\sum_{k=0}^{\infty} a_k (x-a)^k$ converges, we just

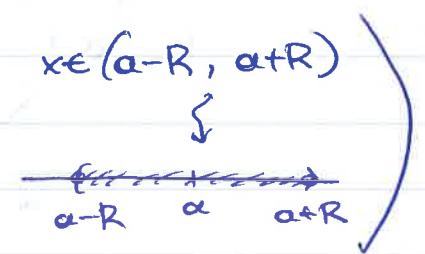
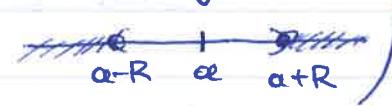
substitute $y = x-a$, we study the convergence behaviour of $\sum_{k=0}^{\infty} a_k y^k$. Indeed, this will yield

some $R \geq 0$ s.t. :

- $\sum_{k=0}^{\infty} a_k y^k$ converges absolutely for $|y| < R$,
- $\sum_{k=0}^{\infty} a_k y^k$ diverges for $|y| > R$,
- $\sum_{k=0}^{\infty} a_k y^k$ demonstrates some special behaviour for $y = R$, $y = -R$.

(4)

Remembering that $y = x - a$, we have that:

- $\sum_{k=0}^{\infty} a_k (x-a)^k$ converges absolutely for $|x-a| < R$
 (i.e. for $x \in (a-R, a+R)$)

- $\sum_{k=0}^{\infty} a_k (x-a)^k$ diverges for $|x-a| > R$
 (i.e. for $x \in (-\infty, a-R) \cup (a+R, \infty)$)

- $\sum_{k=0}^{\infty} a_k (x-a)^k$ demonstrates some special behaviour
 for $x = a-R, x = a+R$.

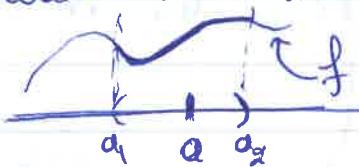
So, hopefully by now we believe that the picture in p.14 of Lecture 3 makes sense.



The reason we actually care so much about power series is that, when they converge, they behave a lot like polynomials (we will explain this more later). Now, polynomials are very easy to handle. So, theoretically, if I wanted to study some function f , and I managed to write it

(5)

as a power series, I would know that it behaves a lot like a polynomial, and hopefully that would give me a lot of insight about f . Even when I want to understand f locally around a point a ,



it similarly makes sense to try and write f as a power series centered at a , in some small interval

(a_1, a_2) containing a . If I could do that, I could pretend that $f(x) = \sum_{k=0}^N a_k (x-a)^k$ (in (a_1, a_2))

for some large N , which is a polynomial centered at a .

actually, the N -th partial sum of the power series.



The reason we are talking about hopefully writing f as a power series

centered at a and in a small interval around a

when we want to study f close to a ,

rather than writing f as a power series centered at 0 on the whole domain of f

is that the first is much more likely to be possible than the second. For instance, remember that

(6)

a power series centered at α may well have a ~~finite interval of convergence~~ symmetric around α ; in which case it doesn't even make sense (as an infinite sum) too far from α (while f may be defined on the whole of \mathbb{R}).

So, if I want to understand f close to α (rather than close to 0) it makes so much more sense to try to write it as $\sum_{k=0}^{+\infty} a_k (x-\alpha)^k$ for x close to α , rather than to write it as $\sum_{k=0}^{+\infty} a_k x^k$ (which would probably not even converge at the point α). At least something of the form $\sum_{k=0}^{+\infty} a_k (x-\alpha)^k$ converges for $x=\alpha$.

You have probably guessed the truth by now: writing a function f as a power series centered at 0 on the whole domain of f may be impossible; even writing it on a given interval I as a power series with center the center of I may be impossible.

Let us see in more detail what actually happens.

(7)

→ Let f be a function, and let $a \in \mathbb{R}$.

Question

If

↑
this is
a BIG if.

f can be written as a power series around a (i.e., if $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ for all x in some interval I containing a),

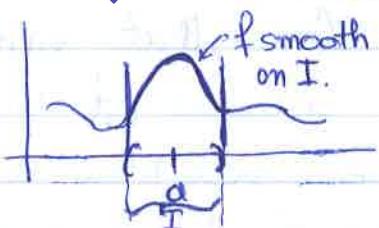
then what are these a_k 's equal to ???

We have an answer in one particular case: when f is smooth around a (i.e., when

there exists some interval I containing a , s.t. f is infinitely differentiable on the interval):

Answer

Let f be a smooth function around a . Then:



if

i.e. on some interval I containing a . there exists a power series

$$\sum_{k=0}^{\infty} a_k (x-a)^k \text{ with}$$

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k \quad \forall x \in I,$$

then

$$a_k = \frac{f^{(k)}(a)}{k!} \quad \forall k=0,1,2,\dots$$

[i.e., then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$ (for $x \in I$).]

(8)

Proof: We have assumed that the series

$\sum_{k=0}^{\infty} a_k(x-a)^k$ converges to $f(x)$ $\forall x \in I$; i.e.:

$$\textcircled{*} \quad f(x) = a_0 + a_1 \cdot (x-a) + a_2 \cdot (x-a)^2 + a_3 \cdot (x-a)^3 + a_4 \cdot (x-a)^4 + \dots$$

for all $x \in I$.

Then, $a_0 = f(a)$. Now, differentiate $\textcircled{*}$ on both sides:

$$\textcircled{*}_1 \quad f'(x) = a_1 + 2 \cdot a_2 (x-a) + 3 \cdot a_3 (x-a)^2 + 4 \cdot a_4 (x-a)^3 + \dots, \quad \forall x \in I$$

So, $a_1 = f'(a)$. Now, differentiate $\textcircled{*}_1$ on both sides:

$$\textcircled{*}_2 \quad f''(x) = 2a_2 + 2 \cdot 3 a_3 (x-a) + 3 \cdot 4 a_4 (x-a)^2 + \dots, \quad \forall x \in I$$

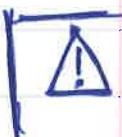
$$\text{So, } 2a_2 = f''(a)$$

$$\Leftrightarrow a_2 = \frac{f''(a)}{2!} \quad \text{Now, differentiate } \textcircled{*}_2 \text{ on both sides:}$$

$$\textcircled{*}_3 \quad f^{(3)}(x) = 2 \cdot 3 a_3 + 2 \cdot 3 \cdot 4 a_4 (x-a) + \dots, \quad \forall x \in I$$

$$\text{So, } 2 \cdot 3 a_3 = f^{(3)}(a)$$

$$\Leftrightarrow a_3 = \frac{f^{(3)}(a)}{3!} \quad \text{And so on...}$$



In each step, we assumed that differentiating our whole



(9)

convergent series is the same as differentiating each term and adding up. Your book tells you that this is always true, but it IS NOT. It is only true under

certain conditions, and the fact that our series is smooth means that the conditions are satisfied. So, we didn't do anything wrong.

→ Def: Let f be a function, smooth on an interval

I, which contains a point $a \in \mathbb{R}$.

We have seen that, if f can be written as a power series centered at a on I , then that power series is $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$.

We call the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$, $\forall x \in \mathbb{R}$, the

Taylor series of f around a .

Note: not I , but \mathbb{R} .

So, the question now is :

Question

When is f equal to its Taylor series around a ? I.e.:
for which $x \in \mathbb{R}$ do we have $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$?

(10)

Answer It depends both on f and the power series.

In particular: For $x \dots$

—... Outside the interval of convergence of $\sum_{k=0}^{+\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$:

We clearly have $\sum_{k=0}^{+\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k \neq f(x)$

for such x , as $\sum_{k=0}^{+\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$ diverges

for such x .

—... Within the interval of convergence of $\sum_{k=0}^{+\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$:

- Obviously $\sum_{k=0}^{+\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$ converges for $x=\alpha$.

- It could be that $f(x) \neq \sum_{k=0}^{+\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$

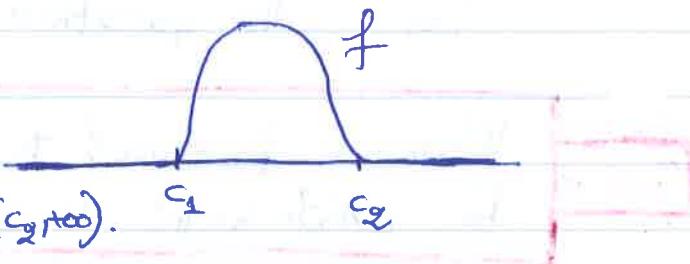
for some x in this interval, and $f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$

for some other x in this interval, with no specific pattern about what happens.

① → A classic example is the following:

Consider the following bump function: it

is smooth, and 0 on $(-\infty, c_1] \cup [c_2, +\infty)$.



(10).

Answer It depends both on f and the power series.

In particular: for $x \dots$

-... Outside the interval of convergence of $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$:

We clearly have $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \neq f(x)$

for such x , as $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$ diverges
for such x .

-... Within the interval of convergence of $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$:

- Obviously $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$ converges for $x=a$.

- It could be that $f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$

for some x in this interval, and $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$

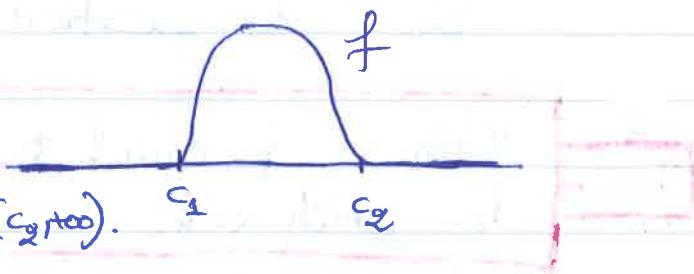
for some other x in this interval, with no specific pattern about what happens.

① → A classic example is the following:

Consider the following

bump function: it

is smooth, and 0 on $(-\infty, c_1] \cup [c_2, \infty)$.



(11)

Let's try to find the Taylor series of f around c_2 :

All the derivatives of f at c_2 are 0, so $\frac{f^{(k)}(c_2)}{k!} = 0 \forall k \geq 1$,

so the Taylor series of f around c_2 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c_2)}{k!} (x - c_2)^k = \sum_{k=0}^{\infty} 0 \cdot (x - c_2)^k \quad \forall x \in \mathbb{R},$$

which equals 0 for all $x \in \mathbb{R}$ (and has infinite radius of convergence, and converges absolutely... all good things.)

So, we have that, for any $x \in \mathbb{R}$,

$$f(x) = \sum_{k=0}^{\infty} 0 \cdot (x - c_2)^k \iff f(x) = 0,$$

which however is not true in (c_1, c_2) ; notice that it is not even true directly left of c_2 , which means that we don't have necessarily that f equals its Taylor expansion around a on a symmetric interval around a ; here we have a lot of asymmetry (equality right of c_2 , not equality directly left of c_2).



If it was true that any function f equals its Taylor expansion around any point a for all $x \in \mathbb{R}$, then, by the above, bump functions wouldn't

(12)

exist! They would have to be 0 on the whole of \mathbb{R} ! So, as convenient as it sounded to have that any function is equal to its Taylor series around any center $\underline{x \in \mathbb{R}}$, it would imply all sorts of weird stuff. Bump functions are very important to define and work on Riemannian manifolds, and we like it that they exist!

(1)

Lecture 5.

2 Sep. 2016.

→ Here is another example, this time of a function

that equals its Taylor series around 0 on the whole interval of convergence of the Taylor series:

Consider $f(x) = \frac{1}{1-x}$, $\forall x \neq 1$ in \mathbb{R} .

We know that $\frac{1}{1-x}$ is the sum of the

series $\sum_{k=0}^{\infty} x^k$, when $|x| < 1$. In other words,

$f(x) = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$. So, $\sum_{k=0}^{\infty} x^k$ is

the Taylor series of f around 0 (we have

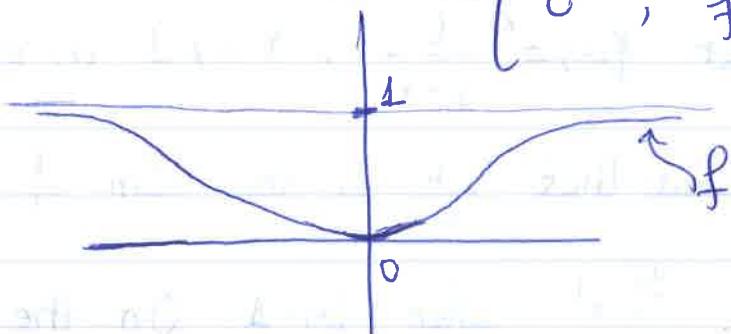
shown that if f , smooth, can be written as a series centered at a locally around a , then that series is the Taylor series around a).

And $f(x) = \sum_{k=0}^{\infty} x^k$ on the whole interval $(-1, 1)$ of convergence of the Taylor series $\sum_{k=0}^{\infty} x^k$.

(2)

→ Here is an example of a function that is equal to its Taylor series around 0 only for $x=0$, while the Taylor series converges on the whole of \mathbb{R} :

Consider $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{for } x \neq 0 \\ 0, & \text{for } x=0 \end{cases}$.



We have that $f(0) = f'(0) = f''(0) = f^{(3)}(0) = \dots = 0$.

So, the Taylor series of f around 0 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} 0 \cdot x^k = 0.$$

But $f(x)=0$ only for $x=0$, which means that f is equal to its Taylor series around 0 only for $x=0$.



Suppose that, $t \geq 0$, $f(t)$ above is the position of an object at time t . This object starts from 0 and, as time passes, it approaches 1. Notice that all of the derivatives of f at 0 are 0; so, at time 0, we have 0 velocity, 0 acceleration,

(3)

0 rate of change of acceleration, etc. And yet, the object moves for $t > 0$! This is not a paradox (just like velocity equal to 0 doesn't mean that the object won't move in the next instant, even though the rate of change of its position at $t=0$ is 0). We should remember that the derivative is an instantaneous quantity: the limit of the rate of change on smaller and smaller intervals. So, a derivative can be 0, even if all these rates of change are positive.

Mainly, the fact that our f has all its derivatives at 0 equal to 0 means that its graph is very, very flat at 0.

→ Eventually: Given a smooth function f and an $a \in \mathbb{R}$, in order to see when we can write f as a power series centered at a , we should:

- ① Find the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$ of f around a . → i.e., find these derivatives.
- ② Find the interval I of convergence of $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$.
- ③ For each $x \in I$, check whether the sum of $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$ equals $f(x)$. It is for the x where this is true that $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$.

(4)

It is this process that we apply to the smooth functions

- $\sin x, x \in \mathbb{R}$
- $\cos x, x \in \mathbb{R}$
- $e^x, x \in \mathbb{R}$
- $\ln(1+x), x > -1$
- $(1+x)^p, x \in \mathbb{R}, \text{when } p \geq 0.$
- $(1+x)^p, x \neq -1, \text{when } p < 0,$
- $\frac{1}{1-x}, |x| < 1.$

and we get that they are equal to their Taylor series in part, or the whole of, their domains:

$$\bullet \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots, \text{ for all } x \in \mathbb{R}.$$

$$\bullet \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots, \text{ for all } x \in \mathbb{R}.$$

$$\bullet e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \text{ for all } x \in \mathbb{R}.$$

$$\bullet \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \text{ for } -1 < x \leq 1.$$

$$\bullet (1+x)^p = \sum_{k=0}^{+\infty} \binom{p}{k} x^k = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots,$$

CHART

Check that for $p \in \mathbb{N}$, $(1+x)^p = \sum_{k=0}^p \binom{p}{k} \cdot x^k$
 (the usual binomial expansion).

for $|x| < 1$.

The last two expressions coincide for $p=-1$ (setting the variable as $-x$ above)

$$\bullet \frac{1}{1-x} = \sum_{k=0}^{+\infty} x^k = 1 + x + x^2 + x^3 + \dots, \text{ for } |x| < 1.$$

A as well as the set of points
 x for which $f(x)$ equals
 the Taylor series! (5)

Knowing these by heart helps us a lot; for instance, we can use them to get Taylor series expansions for other functions, by:

- Adding together: If $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$, $g(x) = \sum_{k=0}^{\infty} b_k (x-a)^k$,
 ↓
 on some interval I
 ↓
 on some interval I'

then $f(x)+g(x) = \sum_{k=0}^{\infty} (a_k + b_k)(x-a)^k$ on at least the intersection of I and I'

- ex: $e^x+x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$
 $+ \quad | \quad | \quad | \quad |$
 $= 1+2x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$,

$\forall x \in \mathbb{R}$

- Multiplying together: If $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$, $g(x) = \sum_{k=0}^{\infty} b_k (x-a)^k$,
 ↓
 on I
 ↓
 on I'

then $f(x) \cdot g(x) = (a_0 + a_1(x-a) + a_2(x-a)^2 + \dots) \cdot (b_0 + b_1(x-a) + b_2(x-a)^2 + \dots)$
 $= a_0 \cdot (b_0 + b_1(x-a) + b_2(x-a)^2 + \dots)$
 $+ a_1(x-a) \cdot (b_0 + b_1(x-a) + b_2(x-a)^2 + \dots)$
 $+ a_2(x-a)^2 \cdot (b_0 + b_1(x-a) + b_2(x-a)^2 + \dots) =$

(6)

$$\begin{aligned}
 &= a_0 b_0 + a_0 b_1 (x-a) + a_0 b_2 (x-a)^2 + a_0 b_3 (x-a)^3 + \dots \\
 &+ a_1 b_0 (x-a) + a_1 b_1 (x-a)^2 + a_1 b_2 (x-a)^3 + \dots \\
 &+ a_2 b_0 (x-a)^2 + a_2 b_1 (x-a)^3 + \dots \\
 &\vdots \\
 &= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x-a) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x-a)^2 + \dots
 \end{aligned}$$

x at least on the intersection
of I and I'

- ex: $\ln(1+x) = x \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \right)$

$$\begin{aligned}
 &+ 1 \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right) \\
 &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots , \quad \underline{-1 < x \leq 1}
 \end{aligned}$$

- Dividing together: If $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ and $g(x) = \sum_{k=0}^{\infty} b_k (x-a)^k$,

we can find the Taylor series around a for $\frac{f(x)}{g(x)}$ by long division of the corresponding series.

The resulting series equals $\frac{f}{g}$ on at least $I \cap I'$,

as long as the zeroes of g cancel the zeroes of f on $I \cap I'$.

(7)

- ex: $\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$

 $\forall x \in \mathbb{R}$

(implying here that
 $\sin x$ is continuously exten-
 ded for $x=0$, to have value 1)

- Substituting one series in another:

If $f(x) = \sum_{k=0}^{+\infty} a_k x^k$,
 ↓
 on I

and $g(x) = \sum_{k=0}^{+\infty} b_k x^k$ on I' ,

and $g(x) \in I \quad \forall x \in I'$, then

$$f(g(x)) = \sum_{k=0}^{+\infty} a_k (g(x))^k = \sum_{k=0}^{+\infty} a_k \left(\sum_{k=0}^{+\infty} b_k x^k \right)^k, \quad \forall x \in I.$$

- ex: $e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \dots =$

$$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots, \quad \forall x \in \mathbb{R}.$$

- Writing a function as an integral of its antiderivative, whose Taylor series we know:

If $f(x) = \int_0^x g(t) dt$, and $g(x) = \sum_{k=0}^{+\infty} a_k x^k$, then

on I

a power series!

$$f(x) = \int_0^x \left(\sum_{k=0}^{+\infty} a_k t^k \right) dt = \sum_{k=0}^{+\infty} a_k \int_0^x t^k dt = \sum_{k=0}^{+\infty} \frac{a_k}{k+1} x^{k+1}, \quad \forall x \in I$$

(8)

$$\begin{aligned}
 -\text{ex: } \arctan x &= [\arctan t]_0^x = \int_0^x \frac{1}{1+t^2} dt = \\
 &= \int_0^x \left(\sum_{k=0}^{+\infty} (-t^2)^k \right) dt \\
 &= \int_0^x \left(1 - t^2 + t^4 - t^6 + t^8 - \dots \right) dt = \\
 &= \int_0^x 1 dt - \int_0^x t^2 dt + \int_0^x t^4 dt - \int_0^x t^6 dt + \int_0^x t^8 dt - \dots \\
 &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots, \text{ for } |x| < 1.
 \end{aligned}$$

$\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = \sum_{k=0}^{+\infty} (-t^2)^k$
 geometric series, for
 $|t^2| < 1 \Leftrightarrow |t| < 1$.

So : Even though the expansions on Chart (8) are few, they can help us find Taylor expansions of many other useful functions f , without having to find $f^{(k)}(a)$ for $k \in \mathbb{N}$, and without having to check if the Taylor series converges to f .



We mentioned earlier that, when a function f can be written as a series around a (i.e., equals its Taylor series around a)

on some interval containing a , then we can pretend

(9)

that f is a polynomial on that interval.

Let's see this in more detail:

first we notice that:

→ If $\sum_{k=0}^{\infty} a_k$ converges to some $s \in \mathbb{R}$,

and s_n is the n -th partial sum of $\sum_{k=0}^{\infty} a_k$,

$$\text{then } R_n := s - s_n \xrightarrow{n \rightarrow \infty} 0$$



this is called
the remainder;
it equals $\sum_{k=n+1}^{\infty} a_k$

Think of the remainder R_n
as the ERROR when you
approximate $\sum_{k=0}^{\infty} a_k$ with
 $\sum_{k=0}^{n-1} a_k$.

Proof: $s_n \xrightarrow{n \rightarrow \infty} s$ (that is what we mean by
" $\sum_{k=0}^{\infty} a_k$ converges to s "),

$$\text{so } s_n - s \xrightarrow{n \rightarrow \infty} 0$$



⚠ The above tells us that the sum of a convergent series can be approximated very well by ^{appropriately} large partial sums. If we apply this for a series $\sum_{k=0}^{\infty} a_k (x-a)^k$, then we get that the power series can be appro-

(1)

ximated well by its partial sums for large n ,
and those are of the form $\sum_{k=0}^{n-1} a_k (x-a)^k$,

i.e.

polynomials!!!

So, morally we can replace a series $\sum_{k=0}^{+\infty} a_k (x-a)^k$

with the polynomial $\sum_{k=0}^N a_k (x-a)^k$ when N is large,

notice
that the
accuracy
of the
approxima-
tion depends
on x .

for all x close to a .

Let's see what we

lose when we do this, in three, quite general, cases:

→ approximating a

power
series

① → Approximation error:

If $f(x) = \sum_{k=0}^{+\infty} a_k x^k$ converges
for $|x| < 1$,

and if $|a_{n+1}| \leq |a_n|$ for $n > N$, then

$$\left| f(x) - \sum_{k=0}^N a_k x^k \right| < \frac{|a_{N+1} x^{N+1}|}{1-|x|}$$

→ the possible
error

If you are interested, the proof is very easy:

$$\left| f(x) - \sum_{k=0}^N a_k x^k \right| = \left| \sum_{k=N+1}^{+\infty} a_k x^k \right| = |a_{N+1}| \cdot |x|^{N+1} + |a_{N+2}| |x|^{N+2} + \dots$$

$\vdots |a_{N+1}| \dots$

\leftarrow terms $|a_n|$ are decreasing

$$\leq |a_{N+1}| \cdot |x|^{N+1} + |a_{N+1}| \cdot |x|^{N+2} + |a_{N+1}| \cdot |x|^{N+3} + \dots$$

$$= |a_{N+1}| \cdot |x|^{N+1} \cdot \underbrace{(1 + |x| + |x|^2 + \dots)}_{\text{geom. series}} =$$

$$= |a_{N+1}| \cdot |x|^{N+1} \cdot \frac{1}{1-|x|}, \text{ for } |x| < 1.$$

(11)



Notice that, when x is very close to 0, then :

- $|x|^{N+1}$ is tiny

- $1 - |x|$ is practically 1, so $\frac{1}{1-|x|}$ is also practically 1

So, when x is very close to 0

(the center of the power series!),

the error is very, very small. So, we can pretend

that $f(x) = \sum_{k=0}^N a_k x^k$, for some large N ,

a polynomial

large enough to make the error so small that we don't care.

②

Approximation error

approximating an

alternating series

Let $\sum_{k=0}^{\infty} a_k$ be an alternating series. If :

$|a_{k+1}| \leq |a_k| \quad \forall k \in \mathbb{N}$

$$a_k \xrightarrow{k \rightarrow \infty} 0$$

in which case the series satisfies the simple test for alternating series, and thus converges to some $s \in \mathbb{R}$

(12)

$$\text{then } |s - \sum_{k=0}^n a_k| \leq |a_{n+1}|.$$

↑ the possible error

↑
the first
term we leave
out of the partial
sum.

⚠ We can of course use this when the alternating series is also a power series! For instance,

the series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, for $-1 < x \leq 1$,

and $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$, for $-1 \leq x \leq 1$,

happen to both satisfy the conditions of the approximation theorem above for

(where the k -th term of each series is

$$a_k = \frac{(-1)^{k+1} x^k}{k}$$
 for the first power series

and $a_k = \frac{(-1)^k x^k}{k!}$ for the second). So, for instance,

- $\ln(1.5)$ can be approximated by $0.5 - \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3}$, with error $\leq \left| \frac{(0.5)^4}{4} \right|$, while

- $e^{-0.5}$ can be approximated by $1 - 0.5 + \frac{(0.5)^2}{2!}$, with

(13)

$$\text{error} \leq \left| \frac{(0.5)^3}{3!} \right|.$$

③

Approximation error:approximating a **function**Let f be a function. Then,

$$\left| f(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k \right| \leq \frac{|x-a|^{n+1} \cdot |f^{(n+1)}(c)|}{(n+1)!},$$

the $(n+1)$ -th
partial sum of
the Taylor series of f

for some $c \in [a, x]$, depending on n .

This theorem helps us as long as $|f^{(n+1)}(c)|$ is quite small; otherwise the error can be very big.
 However, the big advantage of this theorem

is that it tells us how well we can approxi-

mate a **function** with a partial sum of its

Taylor series (i.e. with a polynomial), without knowing a priori that the Taylor series converges to f !

(14).

Notice that the other two approximation theorems approximate convergent power series, which means that we can use them to approximate Taylor series of functions, but, unless we prove that these Taylor series converge to the corresponding functions, we are not allowed to assume that the approximation of the series also approximations of the functions!

(so, for instance, this theorem will be particularly useful in exercise 2 of weekly assignment 2, where we don't know that the Taylor series of $\cos x$ around an angle θ converges to $\cos x$)

→ Ways to use power series

① Summing series: Suppose that we want to

find $\sum_{k=0}^{\infty} a_k$. Can we express this sum as

$\sum_{k=0}^{\infty} b_k (x_0 - a)^k$, for some x_0 , where $\sum_{k=0}^{\infty} b_k (x-a)^k$ is

(15)

a Taylor series of a function f we know? If yes, then $\sum_{k=0}^{\infty} a_k = f(x_0)$.

ex: • $\sum_{k=0}^{\infty} \frac{1}{k!} = ?$ Notice that $\sum_{k=0}^{\infty} \frac{1}{k!}$ is

of $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ equal to the value for $x=1$. So, $\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = e$.

$\underbrace{\hspace{1cm}}_{\text{if } e^x}$

• $\sum_{k=0}^{\infty} \frac{(-3)^k}{k!} = ?$ Notice that $\sum_{k=0}^{\infty} \frac{(-3)^k}{k!}$ is equal to the value of $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x=-3$.

$\underbrace{\hspace{1cm}}_{\text{if } e^x}$

So, $\sum_{k=0}^{\infty} \frac{(-3)^k}{k!} = e^{-3}$.

② Evaluating high order derivatives: Given a function f ,

what is $f^{(n)}(a)$, for some n (any number, maybe large) and $a \in \mathbb{R}$?

If I can expand f as a power series around a , i.e.

if $f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ for x in an interval contain-

(16)

ning a , then we know that $a_k = \frac{f^{(k)}(a)}{k!}$,

for all $k \in \mathbb{N}$. So, in particular, $\frac{f^{(n)}(a)}{n!} = a_n$,

$$\text{i.e. } f^{(n)}(a) = n! \cdot a_n.$$

ex: • Let $f(x) = \cos x$, $x \in \mathbb{R}$. What is $f^{(101)}(0)$?

We know that $\cos x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \forall x \in \mathbb{R}$,

so $f^{(101)}(0)$ is $n! \cdot \left(\begin{array}{l} \text{the coefficient of} \\ x^{101} \text{ in the above} \\ \text{expansion} \end{array} \right)$,

which is 0, as $\cos x$ doesn't have any odd powers in its Taylor expansion around 0.

- Let $f(x) = \frac{\ln(1+x)}{x}$, $x > -1$. Can you find $f^{(5)}(0)$?

③ Solving differential equations :

B1 If f is involved in a differential equation, it may be much easier to solve the equation if we replace f with a partial sum of its Taylor series (as long as

(17)

the partial sum approximates f well). This can help because it is easy to differentiate/integrate polynomials.

ex: Consider

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \cdot \sin\theta \quad (*)$$

the differential

equation describing the motion of a pendulum.

(the 1st partial sum
of the Taylor series of
 $\sin\theta$ around 0.)

If we replace $\sin\theta$ by θ for time t close to 0, which we can check is a good approximation of $\sin\theta$ for θ small, then we can solve much more easily

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \cdot \theta \quad (*)_1$$

approximate the solution of $(*)$ with the solution of $(*)_1$ for θ small.

[3ii]

We can differentiate $(*)$ recursively, to get all the derivatives of the solution f of $(*)$ at 0; so, we can get the Taylor series of f around 0, and use the third approxima-

tion theorem to approximate f by an appropriate partial sum of its Taylor series! So, we can get an approximation of the solution, and its Taylor series.

for instance, if I am given $\theta(0)$, $\theta'(0)$, then,

from $\textcircled{4}$ (which says that $\theta''(t) = -\frac{g}{l} \sin \theta$)

I get $\theta''(0) = -\frac{g}{l} \cdot \sin \theta(0)$,

then I differentiate $\textcircled{4}$ to get $\boxed{\theta'''(t) = -\frac{g}{l} \cdot \cos \theta}$

so $\theta'''(0) = -\frac{g}{l} \cdot \cos \theta(0)$,

then I differentiate $\textcircled{4}$ to get $\theta^{(4)}(t) = +\frac{g}{l} \sin \theta$,

so $\theta^{(4)}(0) = -\frac{g}{l} \cdot \sin \theta(0)$, etc.

So, the Taylor series of the solution of $\textcircled{4}$ around 0 is:

$$\sum_{k=0}^{\infty} \frac{\theta^{(k)}(0)}{k!} x^k, \text{ for } \theta^{(k)}(0) \text{ as above, } k \in \mathbb{N}.$$

④ Approximating integrals: Suppose that $\int_0^x f(t) dt$ is

hard to calculate. If I can write f as a power series, then I can write $\int_0^x f(t) dt$ as a power series too, so I can approximate its value at x with appropriate partial sums.

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ex: $\int_0^x \sin t^2 dt$ is called a fresnel integral,

and appears in optics. It is hard to calculate.

So, we remember that

$$\begin{aligned}\sin(t^2) &= (t^2) - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} - \frac{(t^2)^7}{7!} + \dots \\ &= t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \dots\end{aligned}$$

$$\text{so } \int_0^x \sin t^2 dt = \frac{x^3}{3} - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \dots$$

a series which we can approximate with appropriate partial sums as well as we want (by the error theorem for alternating series). This will probably manage to give us much better approximations than 3i.